JOURNAL OF NUMBER THEORY 6, 1–6 (1974)

On the Analytic Continuation of a Certain Dirichlet Series*

CHUNGMING AN†

Department of Mathematics, Johns Hopkins University, Baltimore, Maryland 21218

Communicated by K. Mahler

Received September 26, 1970

A Dirichlet series associated with a positive definite form of degree δ in n variables is defined by

$$D_F(s, \rho, \alpha) = \sum_{\alpha \in Z^n - \{0\}} F(\alpha)^{-s} e(\rho F(\alpha) + \langle \alpha, \alpha \rangle),$$

where $\rho \in \mathbb{Q}$, $\alpha \in \mathbb{Q}^n$, $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$, $e(a) = \exp(2\pi i a)$ for $a \in \mathbb{R}$, and $s = \sigma + ti$ is a complex number. The author proves that: (1) $D_F(s, \rho, \alpha)$ has analytic continuation into the whole s-plane, (2) $D_F(s, \rho, \alpha)$, $\rho \neq 0$, is a meromorphic function with at most a simple pole at $s = n/\delta$. The residue at $s = n/\delta$ is given explicitly. (3) $\rho = 0$, $\alpha \notin \mathbb{Z}^n$, $D_F(s, 0, \alpha)$ is analytic for $\alpha > n/(\delta - 1)$.

1. Let $F(X) = F(X_1, \ldots, x_n)$ be a homogeneous polynomial of degree δ with integer coefficients such that $F(x) > 0$ for all nonzero $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For any real number ρ and $\alpha = (\alpha_1, \ldots, a_n) \in \mathbb{R}^n$, we shall consider the Dirichlet series of the following type:

$$D_F(s, \rho, \alpha) = \sum_{\gamma \in \mathbb{Z}^n - \{0\}} F(\gamma)^{-s} e(\rho F(\gamma) + \langle \alpha, \gamma \rangle)$$

where $\langle x, y \rangle = x_i y_1 + \cdots + x_n y_n$, $e(a) = \exp(2\pi i a)$ for $a \in \mathbb{R}$ and $s = \sigma + ti$ is a complex number.

From [1][1], we know that $D_F(s, \rho, \alpha)$ converges absolutely and uniformly for $\sigma > \sigma_0 = n/\delta$. Particularly, when $(\rho, \alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^{n+1}$, it is proved in [1] that $D_F(s, \rho, \alpha)$ possesses an analytic continuation into the whole s-plane. In this paper, we shall treat two further cases by two different methods. Without loss of generality, we may assume $0 \leqslant \rho < 1$, $0 \leqslant \alpha_\ell < 1$ for $\ell = 1, \ldots, n$, and one of the α_ℓ's is nonzero.

* This research supported in part by NSF Grant GP-14397.

† Present address: Department of Mathematics, Seton Hall University, South Orange, New Jersey 07079.

[1] The results appeared in Bulletin of AMS, May 1969.

THEOREM. (a) *If ρ and all α_ℓ's are rational numbers, then $D_F(s, \rho, \alpha)$ can be continued analytically into the whole s-plane as a meromorphic function with at most a simple pole at $s = \sigma_0$ with residue*

$$\operatorname*{Res}_{s=\sigma_0} D_F(s, g, \alpha) = (2\pi)^{\sigma_0} \Gamma(\sigma_0)^{-1} G_F(\rho, \alpha) \int_{\mathbb{R}^n} \exp(-2\pi F(x))\, dx,$$

where $dx = dx_1\, dx_2 \cdots dx_n$ and $G_F(\rho, \alpha)$ is the generalized Gaussian sum defined in Section 3 of this paper.

(b) *If $\rho = 0$, then $D_F(s, 0, \alpha)$ can be continued analytically into the half-plane $\sigma > (n-1)/\delta$ as a holomorphic function.*

2. Let $\phi(x) \in S(\mathbb{R}^n)$, the Schwartz space on $\mathbb{R}^n$, and $\hat{\phi}(x)$, be the Fourier Transform of $\phi(x)$ i.e.,

$$\hat{\phi}(y) = \int_{\mathbb{R}^n} \phi(x)\, e(-\langle x, y\rangle)\, dx.$$

It is known that $\hat{\phi}(x)$ is also a Schwartz function.

The following formula is the so-called Poisson summation formula:

$$\sum_{\gamma \in \mathbb{Z}^n} \phi(x+\gamma) = \sum_{\gamma \in \mathbb{Z}^n} \hat{\phi}(\gamma)\, e(\langle x, \gamma\rangle). \tag{1}$$

The theta function of degree δ is defined by

$$\theta_{F,\alpha}(\tau) = \sum_{\gamma \in \mathbb{Z}^n} \exp(2\pi i(\tau F(\gamma) + \langle \alpha, \gamma\rangle)).$$

It is easy to see that $\theta_{F,\alpha}(\tau)$ is holomorphic for $\tau \in H = \{a + bi \in \mathbb{C}, b > 0\}$, the upper half-plane.

Put

$$\mathscr{D}_F(s, \rho, \alpha) = (2\pi)^{-s} \Gamma(s)\, D_F(s, \rho, \alpha).$$

The following lemma can be proved by using the Mellin transform.

LEMMA 1. *For* $\operatorname{Re}(s) = \sigma > \sigma_0$, $t > 0$, we have

$$\mathscr{D}_F(s, \rho, \alpha) = \int_0^\infty [\theta_{F,\alpha}(\rho + ti) - 1]\, t^{s-1}\, dt,$$

$$\theta_{F,\alpha}(\rho + ti) - 1 = \frac{1}{2\pi i} \int_{\sigma-\infty i}^{\sigma+\infty i} \mathscr{D}_F(s, \rho, \alpha)\, t^{-s}\, ds.$$

3. For rational numbers ρ and α_ℓ, $\ell = 1, \ldots, n$, let q be the integer such that $q\rho$ and all $q\alpha_\ell$ are integers. We define the generalized Gaussian sum by

$$G_{F,\alpha}(\rho) = q^{-n} \sum_{\xi \in (\mathbb{Z}/q\mathbb{Z})^n} \exp(2\pi i(\rho F(\xi) + \langle \alpha, \xi\rangle)).$$

This finite sum depends on the choice of q. Actually, it is easy to prove that the sum is independent of the choice of q.

LEMMA 2. *For $t > 0$, we have*

$$\theta_{F,\alpha}(\rho + ti) = t^{-\sigma_0} G_{F,\alpha}(\rho) \int_{\mathbb{R}^n} \exp(-2\pi F(x))\,dx$$
$$+ t^{-\sigma_0} q^n \sum_{\xi \in (\mathbb{Z}/q\mathbb{Z})^n} \Big[e(\rho F(\xi) + \langle \alpha, \xi \rangle) \sum_{\eta \in \mathbb{Z}^n - \{0\}} e\left(\left\langle \frac{\xi}{q}, \eta \right\rangle\right)$$
$$\times \int_{\mathbb{R}^n} \exp(-2\pi F(x))\, e(-\langle x, t^{-1/\delta} q^{-1} \eta \rangle)\,dx \Big].$$

Proof. For $\gamma \in \mathbb{Z}^n$, we write $\gamma = \xi + q\eta$ when $\xi = (\xi_1, \ldots, \xi_n)$, $0 \leqslant \xi_i < q$ and $\eta \in \mathbb{Z}^n$. Then $F(\gamma) = F(\xi) + Mq$, M being an integer which depends only on ξ and η. Hence, for $\tau = \rho + ti$,

$$e(\tau F(\gamma) + \langle \alpha, \gamma \rangle) = e(\rho F(\xi) + \langle \alpha, \xi \rangle)\, e(itF(\gamma)).$$

Thus,

$$\theta_{F,\alpha}(\tau) = \sum_{\xi \in (\mathbb{Z}/q\mathbb{Z})^n} e(\rho F(\xi) + \langle \alpha, \xi \rangle) \sum_{\eta \in \mathbb{Z}^n} e(itF(\xi + qn)).$$

Put $\phi(x) = e(itF(\xi + qx)) \in S(\mathbb{R}^n)$ for each $t > 0$ and ξ. Then

$$\hat{\phi}(y) = \int_{\mathbb{R}^n} \exp(-2\pi t F(\xi + qx) \cdot e(-\langle x, y \rangle)\,dx.$$

From (1) and changing variables by $\xi + qx \to t^{-1/\delta} x$, we shall obtain Lemma 2.

4. We shall prove part (a) of the theorem. By Lemma 1, we may write

$$\mathscr{D}_F(s, \rho, \alpha) = \int_0^1 [\theta_{F,\alpha}(\rho + ti) - 1]\, t^{s-1}\,dt + \int_1^\infty [\theta_{F,\alpha}(\rho + ti) - 1]\, t^{s-1}\,dt.$$

First, we shall show that $I_1(s) = \int_1^\infty (\theta_{F,\alpha}(\rho + ti) - 1)\, t^{s-1}\,dt$ is an entire function of s. To prove this, it is enough to show that for any real number k, $I_1(s)$ converges absolutely and uniformly for $\sigma < k$.

Let $N(n) = \{\gamma \in \mathbb{Z}^n; F(\gamma) = n\}$ and $b_n = \sum_{\gamma \in N(n)} e(\langle r, \alpha \rangle + \rho n)$. Then

$$D_F(s, \rho, \alpha) = \sum_{n=1}^\infty b_n n^{-s}.$$

Since $D_F(s, \rho, \alpha)$ converges absolutely for $\sigma > \sigma_0$, we see that $| b_n | \leqslant$

An^c for some $A > 0$ and $c > \sigma_0$. For $u \geqslant 0$ and any integer N, we have $N! \exp(u) \geqslant u^N$. Putting $N > \max(k, c + 1)$, we obtain, for $t > 0$,

$$\begin{aligned} | \theta_{F,\rho}(\rho + ti) - 1 | \, t^{\sigma-1} &\leqslant A \sum_{n=1}^{\infty} n^c \cdot \exp(-2\pi tn) \, t^{k-1} \\ &\leqslant A(N!)(2\pi)^{-N} \, t^{k-N-1} \sum_{n=1}^{\infty} n^{-(N-c)}. \end{aligned}$$

It is clear to see that $I_1(s)$ converges absolutely and uniformly for $\sigma < k$. Since k can be arbitrarily large ($k \to \infty$), we obtain that $I_1(s)$ is an entire function.

Let, for $\sigma > \sigma_0$,

$$\begin{aligned} I(s) &= \int_0^1 (\theta_{F,\alpha}(\rho + ti) - 1) \, t^{s-1} \, dt \\ &= \int_0^1 \theta_{F,\alpha}(\rho + ti) \, t^{s-1} \, dt - \frac{1}{s}. \end{aligned}$$

By Lemma 2, we get

$$\begin{aligned} I_2(s) &= \int_0^1 \theta_{F,\alpha}(\rho + ti) \, t^{s-1} \, dt \\ &= \int_0^1 G_{F,\alpha}(\rho) \, t^{s-\sigma_0-1} \int_{\mathbb{R}^n} \exp(-2\pi F(x)) \, dx \, dt + I_3(s) \\ &= \frac{1}{s - \sigma_0} G_{F,\alpha}(\rho) \int_{\mathbb{R}^n} \exp(-2\pi F(x)) \, dx + I_3(s), \end{aligned}$$

where

$$\begin{aligned} I_3(s) = \int_0^1 \Big[q^{-n} &\sum_{\xi \in (\mathbf{Z}/q\mathbf{Z})^n} e(\rho F(\xi) + \langle a, \xi \rangle) \sum_{\eta \in \mathbf{Z}^n - \{0\}} e\left(\left\langle \frac{\xi}{q}, \eta \right\rangle\right) \\ &\int_{\mathbb{R}^n} \exp(-2\pi F(x)) \, e(-\langle x, t^{-1/\delta} q^{-1} \eta \rangle) \, dx \Big] \, t^{s-\sigma_0-1} \, dt. \end{aligned}$$

Thus, it is enough to show that $I_3(s)$ is an entire function. Namely, we only need to show that $I_3(s)$ converges absolutely and uniformly for $\sigma > k$, k being any negative real number.

Put $\Psi(x) = \exp(-2\pi F(x)) \in S(\mathbb{R}^n)$. Since $\Psi(x) \in S(\mathbb{R}^n)$, we shall see that for any positive integer N there is a positive constant B depending on N and Ψ such that $| y |^{2N} \, | \Psi(y)| \leqslant B$ for all $y \in \mathbb{R}^n$ and $| y | = \langle y, y \rangle$

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \exp(-2\pi F(x) \, e(-\langle x, t^{-1/\delta} q^{-1} \eta \rangle) \, dx \right| &= | \hat{\Psi}(t^{-1/\delta} q^{-1} \eta)| \\ &\leqslant B t^{2N/\delta} q^{2N} \, | \eta |^{-2N}. \end{aligned}$$

We choose N such that $N > \max((1/2)(\sigma_0 - k)\delta, (1/2)n)$

$$\begin{aligned} | I_3(s)| &\leqslant \int_0^1 q^{-n} \sum_{\xi\in(\mathbb{Z}/q\mathbb{Z})^n} \sum_{\eta\in\mathbb{Z}^n-\{0\}} Bq^{2N} t^{\sigma+(2N/\delta)-\sigma_0-1} | \eta |^{-2N}\, dt \\ &= Bq^{2N} \frac{1}{\sigma + (2N/\delta) - \sigma_0} \sum_{\eta\in\mathbb{Z}^n-\{0\}} | \eta |^{-2N}. \end{aligned}$$

The sum on the left side is the Epstein Zeta function which converges for $N > (1/2)n$. It follows that $I_3(s)$ is an entire function.

What we have proved is the following: for $\sigma > \sigma_0$,

$$\mathscr{D}_{F,\alpha}(s, \rho, \alpha) = I_1(s) + I_3(s) - \frac{1}{s} + \frac{1}{s - \sigma_0} G_{F,\alpha}(\rho) \int_{\mathbb{R}^n} \exp(-2\pi F(x))\, dx,$$

where $I_1(s)$ and $I_3(s)$ are entire functions. Part (a) of the theorem will follow from the above formula.

5. We shall prove part (b) of the theorem.[2] We may, without loss of generality, suppose that $0 < \alpha_1 < 1$ and denote $\bar{x} = (x_2, ..., x_n) \in \mathbb{R}^{n-1}$ for $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

For $\sigma > \sigma_0$, we write $D_F(s, 0, \alpha)$ in the following way.

$$D_F(s, 0, \alpha) = \sum_{\gamma_1\in\mathbb{Z}-\{0\}} F(\gamma_1, \bar{0})^{-s}\, e(\langle\alpha, \gamma\rangle) + \sum_{.\in\mathbb{Z}^{n-1}-\{0\}} \sum_{\gamma_1\in\mathbb{Z}} F(\gamma)^{-s}\, e(\langle\alpha, \gamma\rangle). \tag{2}$$

The first sum in (2) converges absolutely and uniformly for $\sigma > 1/\delta$ (see [1]). Hence it defines a holomorphic function of s for $\sigma > 1/\delta$. The second sum in (2) again defines a holomorphic function for $\sigma > n/\delta$. To obtain the analytic continuation in a large plane, we shall show that the second series converge uniformly when s lies in a compact set in the half-plane $\sigma > (n - 1)/\delta$.

Let us define, for any integer m, $C_m = e(m)/(1 - e(\alpha_1))$. Then $e(m\alpha_1) = C_{m+1} - C_m$. Moreover, $| C_m | = | 1 - e(\alpha_1)|^{-1} = K_2$. We also observe that there are positive constants K_3 and K such that $K^{-1} | x |^{\delta} \leqslant | F(x)| \leqslant K_3 | x |^{\delta}$ and $|(\partial/\partial x_1) F(x)| \leqslant K_3 | x |^{\delta-1}$.

Now, the second series in (2) is majorized by

$$\begin{aligned} &\sum_{|\bar{\gamma}|\neq 0} \sum_{\gamma_1\in\mathbb{Z}} F(\gamma)^{-s}\, e(\langle\alpha, \gamma\rangle) \\ &\quad = \sum_{|\bar{\gamma}|\neq 0} e(\langle\bar{\alpha}, \bar{\gamma}\rangle) \sum_{m\in\mathbb{Z}} (C_{m+1} - C_m) F(m, \bar{\gamma})^{-s} \\ &\quad = \sum_{|\bar{\gamma}|\neq 0} e(\langle\bar{\alpha}, \bar{\gamma}\rangle) \sum_{m\in\mathbb{Z}} C_{m+1}(F(m, \bar{\gamma})^{-s} - F(m + 1, \bar{\gamma})^{-s}). \end{aligned}$$

[2] The case for $n = \delta = 2$ appears in [2].

Furthermore,

$$
\begin{aligned}
&| F(m, \bar{\gamma})^{-s} - F(m+1, \bar{\gamma})^{-s} | \\
&= \left| -s \int_m^{m+1} F(t, \bar{\gamma})^{-s-1} \frac{\partial}{\partial t} F(t, \bar{\gamma})\, dt \right| \\
&\leqslant | s | \int_m^{m+1} K \,|(t, \bar{\gamma})|^{-\delta(\sigma+1)} K_3 \,|(t, \bar{\gamma})|^{\delta-1}\, dt \\
&= | s | \, KK_3 \int_m^{m+1} (t^2 + \gamma_2^2 + \cdots + \gamma_n^2)^{-1/2(\delta\sigma+1)}\, dt.
\end{aligned}
$$

Thus, after changing the variable by $t \to t \mid \bar{\gamma} \mid$, the second series in (2) is majorized by

$$
\begin{aligned}
&| s | \, KK_2K_3 \sum_{|\bar{\gamma}| \neq 0} \int_{-\infty}^{\infty} (t^2 + \gamma_2^2 + \cdots + \gamma_n^2)^{-1/2(\delta\sigma+1)}\, dt \\
&= 2 \, | s | \, KK_2K_3 \sum_{|\bar{\gamma}| \neq 0} | \bar{\gamma} |^{-\delta\sigma} \int_0^{\infty} (1 + t^2)^{-1/2(\delta\sigma+1)}\, dt.
\end{aligned}
$$

Since the above integral converges for $\sigma > 1/\delta$, we see that it has the majorant $K_4 \sum_{|\bar{\gamma}| \neq 0} | \bar{\gamma} |^{-\delta\sigma}$, where K_4 depends only on α, F, and the compact set in which s lies.

We observe that the series $D_F(s, 0, \alpha)$ summed in this particular manner converges uniformly when s lies in a compact set in the half-plane $\sigma > (n-1)/\delta$. By Weierstrass' theorem, it provides the necessary analytic continuation into this large half-plane.

Acknowledgment

The methods used to prove part (a) of the theorem are due to Professor T. Ono. He gave the definition of the Gaussian sum $G_F(\rho, 0)$ and the analytic continuation into the half-plane $\sigma > 0$ for the case $\alpha = 0$, with ρ rational, in a lecture he gave at Johns Hopkins University. I modified the definition of Gaussian sum and the method for nonzero α. I thank him for suggesting this problem.

References

1. Chungming An, On a generalization of the gamma function and its application to certain dirichlet series, Dissertation, University of Pennsylvania, 1969.
2. C. L. Siegel, Lecture on advanced analytic number theory, Tate Institute of Fundamental Research, Bombay, India, 1961.